Modes of Propagation in a Coaxial Waveguide with Lossless Reactive Guiding Surfaces

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Abstract—An analysis of the modes that can propagate in a coaxial waveguide with lossless reactive guiding surfaces is presented. The cases where both the surfaces are either inductive or capacitive and the case where one of the surfaces is capacitive and the other is inductive are discussed. The results show that, in general, there are two surface waves and an infinite number of waveguide modes. Whereas all the waveguide modes show the cut off phenomenon, the surface waves may either propagate down to zero frequency or get transformed into the lowest order waveguide mode at certain critical frequencies determined by the structure parameters.

I. Introduction

After Barlow's suggestion of screening a surface waveguide [1]–[3], the first systematic study of the modes existing in a parallel-plate waveguide with inductive surfaces was made by Wait [4]. This was extended by Arora and Vijayaraghavan [5] to cover the cases where the supporting surfaces are both capacitive or where one of them is inductive and the other is capacitive. The present investigation of the modes of propagation in a coaxial waveguide is similar to the one already carried out for the parallel-plate waveguide and covers identical guide-surface reactance possibilities.

It is well known that a thin coating of dielectric on a conductor enhances the inductive reactance, and several practical arrangements have been discussed by Barlow [2], [3]. However, realization of a capacitive reactance is less straightforward as such surfaces need thicker dielectric coatings. Besides, the capacitive reactance depends more strongly on the type of field that exists over the surface. Despite these limitations, the concept of surface impedance has proved to be a very useful one and is widely used in the literature. In the analysis to follow, it is assumed for simplicity that any desired reactance, inductive or capacitive, can be realized, the means of realization being secondary, and that the reactance is constant whatever be the propagating mode or modes.

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II. Modal Analysis

Consider an infinite coaxial waveguide with its axis aligned along the z axis of the circular cylindrical coordinate system defined by the coordinates r, θ, z. The inner and outer guiding surfaces are $r=a$ and $r=b$ with surface impedances Z_a and Z_b, respectively. The analysis will be restricted to axially symmetric modes propagating along the positive z direction. Asymmetric modes can, of course, be excited, but such modes are of the hybrid type and there are added complexities in their analysis. In particular, hybrid modes require the specification of two values of surface reactance for each surface [6]. In view of these complexities, such modes will not be considered here but will form the subject of a subsequent study. In the analysis to follow, a time dependence of the form $e^{j\omega t}$ will be implicit.

Consider first TM modes having field components E_r, H_θ, E_z, of which the components E_r and E_z can be expressed in terms of H_θ by the equations

$$E_z = (1/j\omega\epsilon_0 r)\partial(rH_\theta)/\partial r$$

$$E_r = -(1/j\omega\epsilon_0)\partial H_0/\partial z \qquad (1)$$

while H_θ itself can be determined by solving the wave equation

$$\frac{\partial^2 H_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial H_\theta}{\partial r} + \frac{\partial^2 H_\theta}{\partial z^2} + \left(k_0^2 - \frac{1}{r^2}\right)H_\theta = 0 \qquad (2)$$

subject to the boundary conditions

$$E_z/H_\theta = Z_a, \qquad \text{at } r = a$$

$$E_z/H_\theta = -Z_b, \qquad \text{at } r = b. \qquad (3)$$

In (2), $k_0^2 = \omega^2\mu_0\epsilon_0$.

The solution of (2) may be expressed in a series form comprising all the possible modes of propagation:

$$H_\theta = \sum_n \left[A_n I_1(u_n r) + B_n K_1(u_n r)\right]e^{-j\beta_n z} \qquad (4)$$

where A_n and B_n are hitherto undetermined amplitude constants, I_1 and K_1 denote modified Bessel functions of the first order, β_n is the longitudinal phase constant, and u_n is the transverse propagation constant related to β_n by the equation $u_n = \sqrt{\beta_n^2 - k_0^2}$. Real values of u_n

specify surface wave modes and their imaginary values the waveguide modes.

Relations (4), (1), and (3), when combined, restrict H_θ to the form

$$H_\theta = \sum A_n \left[I_1(u_n r) + \frac{u_n I_0(u_n a) - j\omega\epsilon_0 Z_a I_1(u_n a)}{u_n K_0(u_n a) + j\omega\epsilon_0 Z_a K_1(u_n a)} K_1(u_n r) \right] e^{-j\beta_n z} \quad (5)$$

with u_n determined by the characteristic equation

$$\frac{u_n + j\omega\epsilon_0 Z_a \dfrac{K_1(u_n a)}{K_0(u_n a)}}{u_n - j\omega\epsilon_0 Z_a \dfrac{I_1(u_n a)}{I_0(u_n a)}} \; \frac{u_n + j\omega\epsilon_0 Z_b \dfrac{I_1(\alpha u_n a)}{I_0(\alpha u_n a)}}{u_n - j\omega\epsilon_0 Z_b \dfrac{K_1(\alpha u_n a)}{K_0(\alpha u_n a)}} = \frac{I_0(u_n a) K_0(\alpha u_n a)}{I_0(\alpha u_n a) K_0(u_n a)} \quad (6)$$

in which $\alpha = b/a$, the ratio of the radii.

In a similar way it may be shown that the E_θ component of a TE field (components E_θ, H_r, H_z) in the same structure can be expressed in a series form identical with that in (5), with $\epsilon_0 Z_a$ replaced by $1/\mu_0 Z_a$ and $\epsilon_0 Z_b$ by $1/\mu_0 Z_b$. The characteristic equation for this case is obtained from (6) by making the same substitutions therein.

A. Inductively Reactive Surfaces

1) *Transverse Magnetic Modes:* Let the guiding surfaces be inductively reactive with $Z_a = jX_a$, $Z_b = jX_b$, and X_a, $X_b > 0$. Then (6) becomes

$$\frac{U - P\dfrac{K_1(U)}{K_0(U)}}{U + P\dfrac{I_1(U)}{I_0(U)}} \; \frac{U - Q\dfrac{I_1(\alpha U)}{I_0(\alpha U)}}{U + Q\dfrac{K_1(\alpha U)}{K_0(\alpha U)}} = \frac{I_0(U)}{I_0(\alpha U)} \; \frac{K_0(\alpha U)}{K_0(U)} \quad (7)$$

where $U = u_n a$, $P = \omega\epsilon_0 X_a a$, and $Q = \omega\epsilon_0 X_b a$ (all dimensionless quantities). Real roots of (7) signify surface wave modes and the imaginary roots the waveguide modes.

Consider, first, the real roots of the equation. From comparison with the corresponding parallel-plate problem [5], it may be anticipated that this equation would have two nontrivial real roots. A detailed graphical and numerical analysis [7] shows that this indeed is so and that one of these roots vanishes when the quantity P is decreased below a certain critical value governed by the ratio Q/P, indicating that the corresponding wave, hereafter called type I surface wave, loses its surface wave character at that critical value of P. The other root, however, remains nonzero for all P except $P = 0$ when it vanishes revealing that the surface wave mode, to be called type II hereafter, cannot be supported by a guide with perfectly conducting guide walls.

The critical value of P for a given Q/P ratio, at which the type I wave loses its surface wave character, can be easily deduced from the characteristic equation (7) by making U approach zero. This yields the condition

$$\frac{1}{P} + \frac{\alpha}{Q} = \frac{1}{2}(\alpha^2 - 1). \quad (8)$$

To find the imaginary roots of (7), set $U = jV$. On this substitution, (7) takes the form

$$\frac{VJ_0(V) + PJ_1(V)}{VN_0(V) + PN_1(V)} = \frac{VJ_0(\alpha V) - QJ_1(\alpha V)}{VN_0(\alpha V) - QN_1(\alpha V)} \cdot \quad (9)$$

This equation is found to have an infinite number of roots corresponding to an infinity of waveguide modes. Of all these, the one that is found to merit attention is the lowest order root, specifying, what is to be called in future, the zero-order waveguide mode. This root is found to vanish as the quantity P is increased above a certain critical value. This critical condition, which may be deduced by letting V tend to zero in (9), is found to be the same as (8). The obvious conclusion to be drawn is that this critical condition marks the transition between the surface wave mode of type I and the zero-order waveguide mode.

Taking $\alpha = 2$ as a typical value, (7) and (9) were solved on the computer for the two real roots and the lowest imaginary root in the range $0 < V < \pi$ for various values of P, with Q/P as a parameter. From the computed values of the real roots, the dispersion curves (βa versus ka) are plotted for the two surface waves in Figs. 1 and 2, taking $X_a = \sqrt{\mu_0/\epsilon_0}$, the characteristic impedance of free space. This choice of X_a makes $P = ka$, and βa is found through the relation $\beta a = \sqrt{V^2 + (ka)^2}$.

Fig. 1 shows the dispersion curves of the surface wave of type I (continuous lines) for various Q/P ratios. The curves are found to terminate on the line $\beta a = ka$ or when U becomes zero. The dotted lines are the dispersion curves for the zero-order waveguide mode obtained from the roots of (9). Fig. 1 clearly shows the smooth transition of the type I surface wave into the zero-order waveguide mode at the critical frequencies determined by the intersections of the $\beta a = ka$ line with the dispersion curves. The points where the dotted curves intersect the ka axis give the cutoff frequencies for the zero-order mode. Fig. 2 displays the dispersion curves for the surface wave mode of type II, showing that this mode propagates down to zero frequency, unlike the type I mode.

2) *Transverse Electric Modes:* An analysis of the characteristic equation for TE modes (refer to (6) and the following paragraph) reveals that the structure cannot support TE surface waves but can support an infinity of TE waveguide modes, all exhibiting the cutoff phenomenon.

In practice, then, choosing an excitation to generate only TM modes at a frequency such as to make the

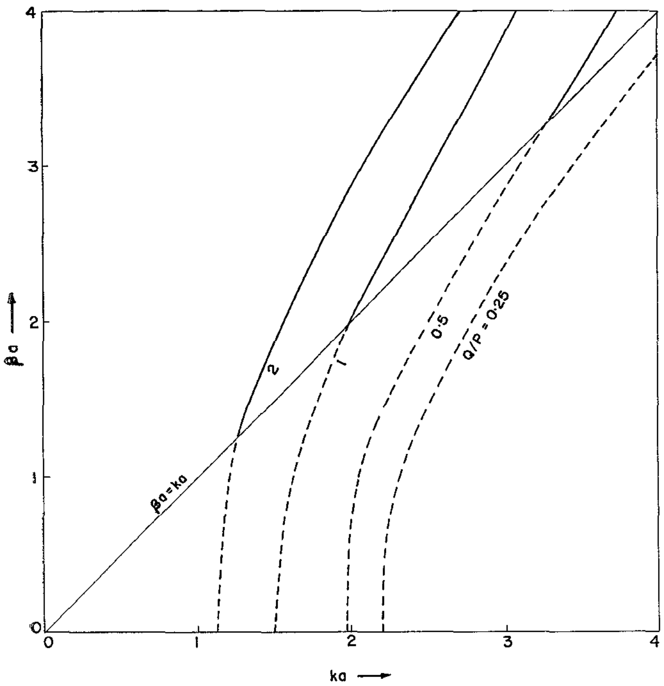

Fig. 1. Dispersion curves for type I surface wave mode (continuous lines) and zero-order waveguide mode (dotted lines) for $X_a=(\mu_0/\epsilon_0)^{1/2}$ and different ratios Q/P.

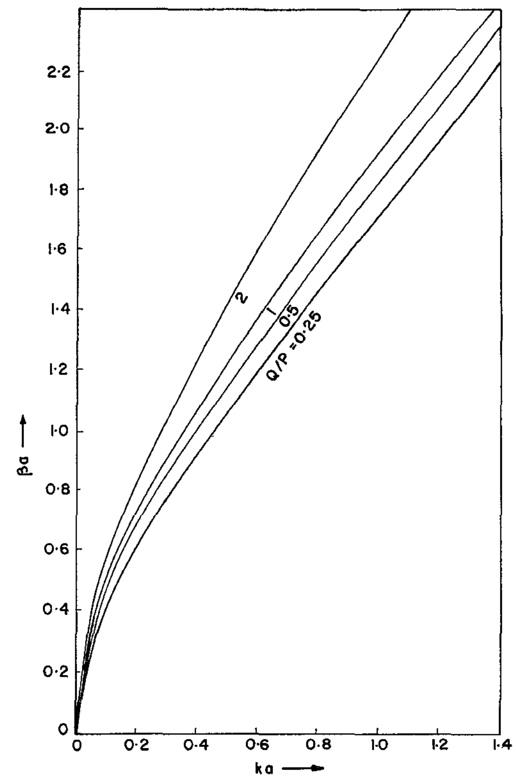

Fig. 2. Dispersion curves for type II surface wave mode for $X_a=(\mu_0/\epsilon_0)^{1/2}$ and different ratios Q/P.

surface wave of type I critical and proportioning the guide so that all waveguide modes are cut off, one can make only the type II surface wave mode propagate without resorting to any special artifice for effecting mode purity. This mode is the same as the hybrid TEM wave of Barlow. Similar results were obtained for the parallel-plate waveguide [5].

B. Capacitively Reactive Surfaces

The results for the case when both surfaces are capacitive are readily deduced from those for the case of inductively reactive surfaces by applying the duality principle. It suffices to say that two TE surface waves and an infinity of TE and TM waveguide modes exist and one of the surface waves transforms with the zero-order TE waveguide mode at a certain critical frequency determined by a relation analogous to (8).

C. One Surface Inductive and the Other Capacitive

The modal analysis for this case of oppositely reactive guiding surfaces proceeds on the same lines as in the case of inductively reactive surfaces. It is sufficient to mention only the results of such an analysis [7].

At the outset one sees two possibilities, the first one where the inner surface is inductive and the outer capacitive and the second one where the inner surface is capacitive and the outer inductive.

1) *Inner Surface Inductive and Outer Surface Capacitive:* When the surface $r=a$ has an inductive reactance X_a and the surface $r=b$ a capacitive reactance $-X_b$, one TM and one TE surface wave can exist along with an infinite number of TM and TE waveguide modes.

The TM surface wave propagates when $1/P-\alpha/Q<(\alpha^2-1)/2$ and goes over into the zero-order TM waveguide mode when $1/P-\alpha/Q>(\alpha^2-1)/2$, where $P=\omega\epsilon_0X_aa$ and $Q=\omega\epsilon_0X_ba$ as before.

For TE modes the corresponding transition occurs when $\alpha/Q'-1/P'=(\alpha^2-1)/2$, where $P'=\omega\mu_0a/X_a$ and $Q'=\omega\mu_0a/X_b$. The mode has surface wave or waveguide character depending, respectively, on whether $\alpha/Q'-1/P'$ is less than or greater than $(\alpha^2-1)/2$.

It may be noted that when $Q/P\leq\alpha$ for the TM case or when $Q'/P'\leq\alpha$ for the TE case, the corresponding surface wave modes do not become critical. The TM surface wave gets associated with the inductive surface and the TE wave with the capacitive surface.

Fig. 3 shows the dispersion curve for the TM surface wave in the structure. The figure clearly brings into focus the critical behavior of the surface wave, and especially the fact that the mode goes critical only for $Q/P\geq\alpha$ ($\alpha=2$ for the calculations). As before, the intersection of the $\beta a=ka$ line with a dispersion curve gives the critical frequency.

2) *Inner Surface Capacitive and Outer Surface Inductive:* The results for this case follow from those derived for the case discussed before through an application of the duality principle. Accordingly, a TM surface wave exists for $\alpha/P-1/Q<(\alpha^2-1)/2$ and goes into the wave

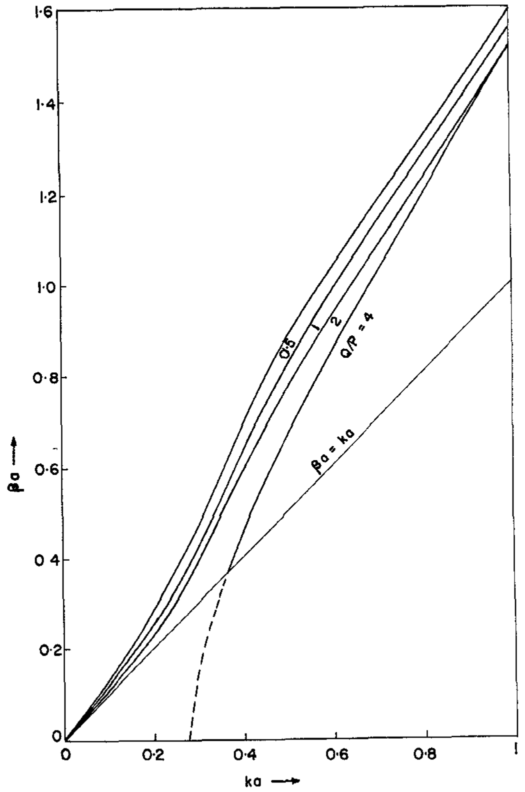

Fig. 3. Dispersion curves for TM surface wave mode (continuous lines) and zero-order waveguide mode (dotted lines) for $X_a = (\mu_0/\epsilon_0)^{1/2}$ and different ratios Q/P when inner surface is inductive and outer surface capacitive.

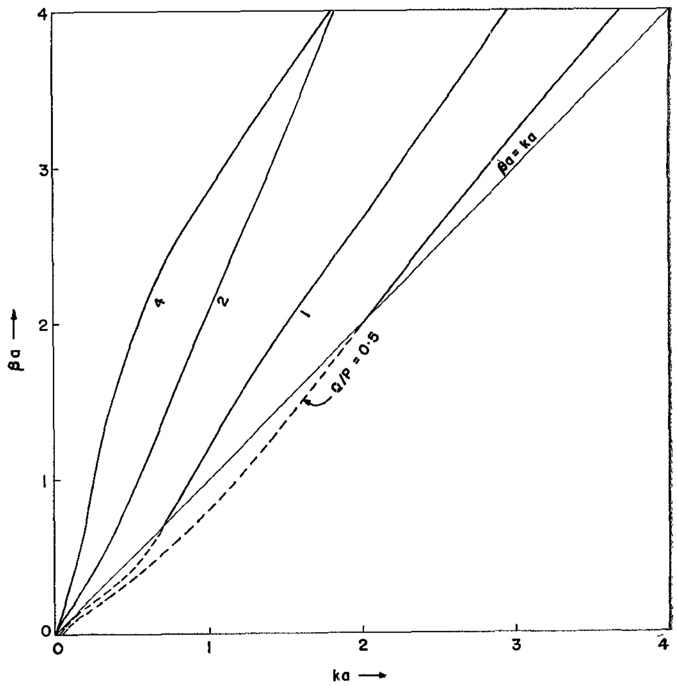

Fig. 4. Dispersion curves for TM surface wave mode (continuous lines) and zero-order waveguide mode (dotted lines) for $X_a = (\mu_0/\epsilon_0)^{1/2}$ and different ratios Q/P when inner surface is capacitive and outer surface inductive.

guide mode when the left-hand side of the inequality exceeds the right-hand side. Similarly a TE surface wave exists for $1/P' - \alpha/Q' < (\alpha^2 - 1)/2$ and goes over into a TE waveguide mode when the left-hand side exceeds the right-hand side. The surface waves do not become critical for $Q/P \geq \alpha$ (TM case) or $Q'/P' \geq \alpha$ (TE case). Fig. 4 shows the dispersion curves of a TM surface wave for this case and illustrates the fact that the wave goes critical for $Q/P \leq \alpha = 2$.

III. Surface Wave Field Patterns for Inductively Reactive Surfaces

The electric field patterns in the θ = constant plane can be computed by solving the differential equation

$$\frac{\text{Re}\,(E_z)}{\text{Re}\,(E_r)} = \frac{dz}{dr}. \tag{10}$$

From (5) and (1), the E_z and E_r components associated with the two surface waves may be stated in the general forms

$$E_z = \frac{Au}{j\omega\epsilon_0}\,[I_0(ur) + CK_0(ur)]e^{-j\beta z} \tag{11a}$$

$$E_r = \frac{A\beta}{\omega\epsilon_0}\,[I_1(ur) - CK_1(ur)]e^{-j\beta z} \tag{11b}$$

where A is an amplitude constant and

$$C = \frac{UI_0(U) + PI_1(U)}{-UK_0(U) + PK_1(U)} \tag{11c}$$

with the appropriate value of u. Substituting (11) into (10) and integrating, one obtains the expression for the field lines in the θ = constant plane:

$$ur[I_1(ur) - CK_1(ur)] \sin \beta z = \text{constant}. \tag{12}$$

The individual patterns for the type I and type II waves are obtained by substituting the appropriate eigenvalues in (12).

For the type I wave (designated by subscript 1), one finds that the E_z component is a maximum at a surface $r = m_1$ for which

$$C = C_1 = \frac{I_1(u_1m_1)}{K_1(u_1m_1)}. \tag{13}$$

Fig. 5 shows the type I wave field which is seen to bear a close resemblance to the lowest order TM mode in a conventional coaxial waveguide with perfectly conducting walls.

For the type II wave (designated by subscript 2), $E_z = 0$ at a surface $r = m_2$ for which

$$C = C_2 = -\frac{I_0(u_2m_2)}{K_0(u_2m_2)}. \tag{14}$$

Fig. 6 shows the field pattern for this wave and one observes that this mode is nothing but the hybrid TEM

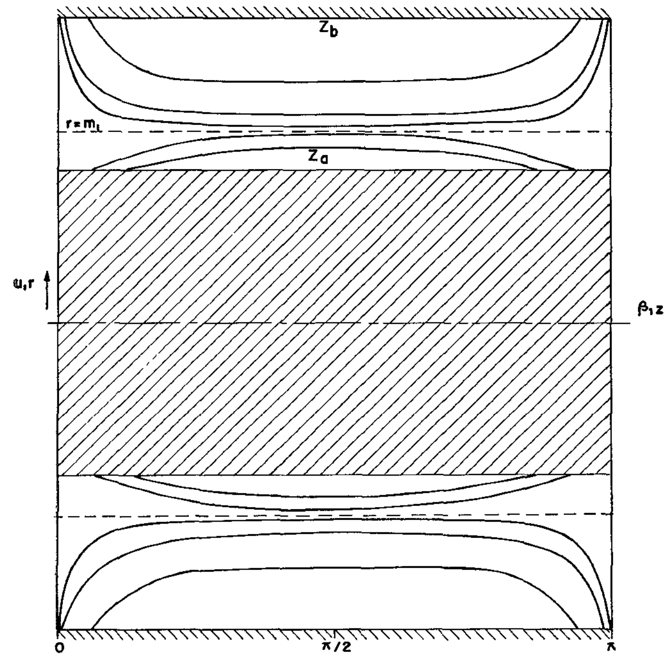

Fig. 5. Electric field pattern for type I surface wave mode.

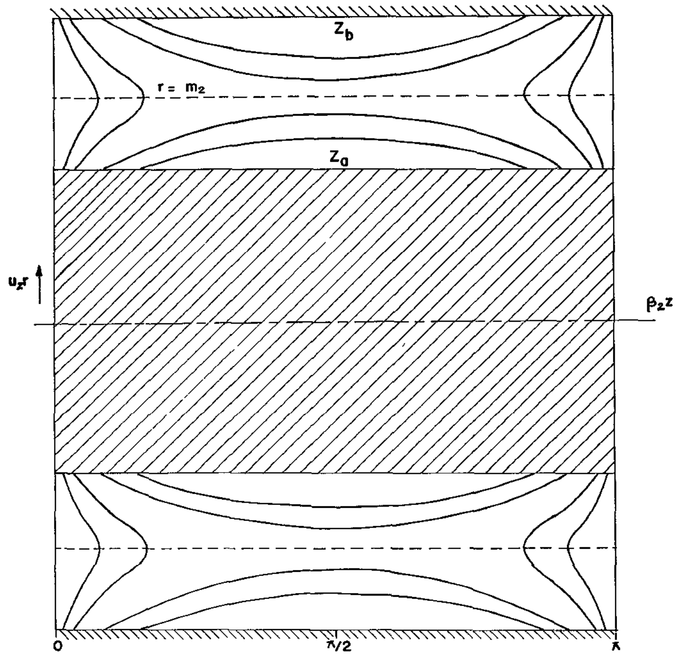

Fig. 6. Electric field pattern for type II surface wave mode.

wave of Barlow. In fact, one can trace the transition of the surface wave mode of type I into the lowest order TM waveguide mode and that of the type II surface wave mode into the TEM wave of a coaxial guide with perfectly conducting walls as the surface reactances are gradually reduced to zero.

IV. Conclusion

A theoretical study of the propagating modes in a coaxial structure with reactive walls yields results similar to those derived from a study of the modes in a corresponding parallel-plate waveguide [5]. In general, there are two surface waves and an infinity of waveguide modes. One of the two surface waves is the hybrid TEM wave propagating down to zero frequency, while the other goes critical and gets transformed into the lowest order waveguide mode at certain frequencies determined by structure parameters. It is possible to proportion the waveguide suitably to effect uncontaminated mode propagation in the hybrid TEM surface wave mode.

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